3) With an increase of the liquid-fuel mass flow and heat conductivity of gases, the evaporation and combustion rates rise, but with an increase of gases' heat capacity, viscosity, and the fuel density, the evaporation and combustion rates decrease.

4) The calculated zone length for the evaporation and combustion of drops of a single-component or a previously intermixed liquid fuel is close to the length of an LRE chamber, as determined by an empirical method.

—*Submitted February 2, 1960*

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Reviewer's Comment

To the best of the reviewer's knowledge, this is the first published Russian analysis of a detailed model of the combustion process in a liquid propellant rocket motor. A steady-state, one-dimensional model involving two-phase flow (liquid droplets and gaseous reaction products) is considered. The distribution in droplet sizes is neglected, and vaporization is assumed to be the rate-controlling process.

The author's study apparently was inspired by the work of Sodha (Ref. 1 of the paper); Il'yashenko appears to be unaware of the more significant publications in English (1–6) which began with the classical analysis of Probert (1) in 1946 and included a wealth of NASA research (5). The author's model resembles that of Spalding (3), but his treatment retains the Lagrangian aspects of Sodha's analysis instead of the simpler and more transparent Eulerian approach of Spalding.

The author's approximations appear to be less realistic than those of Spalding. The treatment of gas and droplet accelerations is particularly questionable and does not seem to pay proper respect to Newton's second law. It is curious how, while explicitly assuming that the acceleration of the droplets is equal and not opposite to that of the gas, the author finds that the relative velocity of the gas and the droplets decreases as they travel downstream. Nevertheless, the author obtains the usual results, which are virtually unavoidable in one-dimensional analyses (7): the required length of the combustion chamber increases as the injection velocity and droplet diameter increase and decreases as the evaporation rate increases.

It is worth mentioning that the correlation between drag coefficient and Reynolds number used by the author (Eq. [6]) was obtained from early Russian work (Ref. 4 of the paper) and appears to be less accurate than the correlations currently in use (6) in this country. Another interesting observation is that the only data quoted by the author in his experimental comparison is for the German V-2 and the Rocketdyne F-1.

Since the author's analysis is second-rate by Western standards, we may conclude that either 1) other, more realistic analyses exist in Russia or 2) successful rocket motor design does not require analyses of this kind.

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JANUARY 1963 AIAA JOURNAL VOL. 1, NO. 1

Equations of the Precessional Theory of Gyroscopes

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EULER'S equation for gyroscopic motion, obtained by using the theorem of moments, is as follows

$$\begin{aligned} A(\dot{p} - qr') + Hq &= M_x \\ A(\dot{q} + pr') - Hp &= M_y \\ \dot{H} &= M_z \end{aligned} \quad [1]$$

where z is the gyroscope axis, x and y are the axes in the equatorial plane of the inertia ellipsoid, constructed for the suspension point; A is the gyroscope's equatorial moment of inertia; H is the angular momentum of the gyroscope; p and q are the projections of the gyroscope's angular velocity on the x and y axes; r' is the projection of the angular velocity for trihedron x, y, z upon the x axis; M_x, M_y, and M_z are the moments of forces about the axes of the moving trihedron.

In the precessional (or elementary) theory, it is assumed that the magnitude of the gyroscope's kinetic moment is approximately equal to H and is directed along the axis of the gyroscope. Then, the following equation is obtained:

$$H_q = M_x \qquad -Hp = M_y \qquad \dot{H} = M_z \quad [2]$$

Let axes ξ, η, and ζ be approximated graphically. The position of the gyroscope's axes in this system of coordinates will be defined by angles α and β as indicated in Fig. 1.

Translated from Uchenye Zapiski (Leningradskogo Universiteta, Seriia Matematicheskikh Nauk (Scientific Notes of Leningrad University, Mathematical Science Series), **35**, no. 280, 25–30 (1960). Translated by Primary Sources, New York.

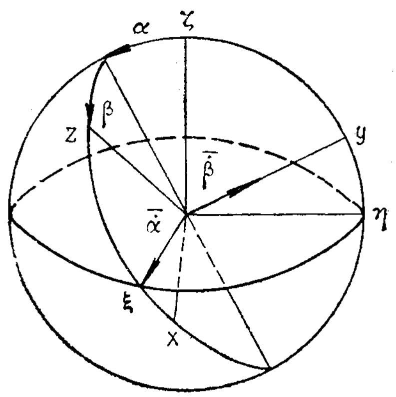

Fig. 1

The component velocities of the angular velocity for trihedron ξ, η, and ζ will be

$$
\begin{aligned}
u_\xi &= -\frac{V_N}{R} \\
u_\eta &= u\cos\varphi + \frac{V_E}{R} \qquad [3] \\
u_\zeta &= u\sin\varphi + \frac{V_E}{R}\tan\varphi
\end{aligned}
$$

Here, u is the angular velocity of the earth's daily rotation; R is the earth's radius; φ is the latitude of the place; V_N and V_E are the northerly and easterly components of the velocity of motion of the suspension point along the surface of the earth.

Let us assume that angles α and β are small. Then

$$p = \dot\alpha + u_\xi - u_\zeta\beta = \dot\alpha - \frac{V_N}{R} - \left[u\sin\varphi + \frac{V_E}{R}\tan\varphi\right]\beta$$

$$q = \dot\beta + u_\eta + u_\zeta\alpha = \dot\beta + u\cos\varphi + \frac{V_E}{R} + \left[u\sin\varphi + \frac{V_E}{R}\tan\varphi\right]\alpha$$

$$r' = u_\xi\beta - u_\eta\alpha + u_\zeta = u\sin\varphi + \frac{V_E}{R}\tan\varphi - \left[\frac{V_N}{R}\beta + \left(u\cos\varphi + \frac{V_E}{R}\right)\alpha\right]$$

There is no doubt that the latter terms in these equations will be small as compared with the first. Hence, the following approximation can be assumed:

$$
\begin{aligned}
p &= \dot\alpha - \frac{V_N}{R} \\
q &= \dot\beta + u\cos\varphi + \frac{V_E}{R} \qquad [4] \\
r' &= u\sin\varphi + \frac{V_E}{R}\tan\varphi
\end{aligned}
$$

Assuming that H is constant and substituting [4] in [1], we will obtain[1]

$$A\left[\ddot\alpha - \frac{\dot V_N}{R} - \left(\dot\beta + u\cos\varphi + \frac{V_E}{R}\right)\left(u\sin\varphi + \frac{V_E}{R}\tan\varphi\right)\right] + H\left(\dot\beta + u\cos\varphi + \frac{V_E}{R}\right) = M_x$$

$$A\left[\ddot\beta + \frac{\dot V_E}{R} + \left(\dot\alpha - \frac{V_N}{R}\right)\left(u\sin\varphi + \frac{V_E}{R}\tan\varphi\right)\right] - H\left(\dot\alpha - \frac{V_N}{R}\right) = M_y$$

The latter terms, enclosed in brackets, are small in comparison with the first terms and can be disregarded. Then

$$
\begin{aligned}
A\ddot\alpha + H\dot\beta &= M_x + \frac{A}{R}\dot V_N - H\left(u\cos\varphi + \frac{V_E}{R}\right) \\
A\ddot\beta - H\dot\alpha &= M_y - \frac{A}{R}\dot V_E - H\frac{V_N}{R}
\end{aligned} \qquad [5]
$$

By substituting [4] in [2] and by identifying α and β with a dot for differentiation purposes, we obtain

$$
\begin{aligned}
H\beta_1 &= M_x - H\left(u\cos\varphi + \frac{V_E}{R}\right) \\
-H\dot\alpha_1 &= M_y + H\frac{V_N}{R}
\end{aligned} \qquad [6]
$$

Let us assume

$$
\begin{aligned}
\alpha + i\beta &= \theta \qquad V_E + iV_N = V \\
\alpha_1 + i\beta_1 &= \vartheta \qquad M_x + iM_y = M
\end{aligned} \qquad [7]
$$

In this case, systems [5] and [6] will become

$$A\ddot\theta - iH\dot\theta = M - i\frac{A}{R}\dot V - \frac{H}{R}V - Hu\cos\varphi \qquad [8]$$

$$-iH\dot\vartheta = M - \frac{H}{R}V - Hu\cos\varphi \qquad [9]$$

If we assume that the gyroscope is definitely symmetrical with respect to angles α and β (which is true in many cases), then M can always be represented as a certain function of θ, $\dot\theta$, and t. The following can be assumed: when angles α and β are small

$$M = m_0 + m_1\theta + m_2\dot\theta \qquad [10]$$

The special solutions of [8] and [9], used to determine the gyroscopic instrument error, are of special interest for the applied theory of gyroscopes. The problem of differentiating between the given special solutions is discussed in this paper for the gyroscopic pendulum.

When resistance is not taken into account for the gyroscopic pendulum, we have

$$m_0 = -\frac{Pl}{g}\dot V_N + i\frac{Pl}{g}\dot V_E = i\frac{Pl}{g}\dot V$$

$$m_1 = -Pl \qquad m_2 = 0$$

where P is the weight of the gyroscope, and l is the distance from its center of gravity to its point of support. Then Eqs. [8] and [9] will have the form

$$A\ddot\theta - iH\dot\theta + Pl\theta = -Hu\cos\varphi - \frac{H}{R}V + i\left(\frac{Pl}{g} - \frac{A}{R}\right)\dot V \qquad [11]$$

$$-iH\dot\vartheta + Pl\vartheta = -Hu\cos\varphi - \frac{H}{R}V + i\frac{Pl}{g}\dot V \qquad [12]$$

Let us note that term A/R is small in comparison with Pl/g and can be disregarded.

[1] It appears that M_x and M_y should also include the components of the inertial forces of the forward translational motion.

The general solutions for Eqs. [11] and [12] will be

$$\theta = c_1 e^{i\omega_1 t} + c_2 e^{i\omega_2 t} - \frac{u}{\omega}\cos\varphi + \frac{e^{i\omega_1 t}}{\omega_1 - \omega_2}\int_0^t e^{-i\omega_1 t}\left(i\frac{H}{AR}V + \frac{Pl}{Ag}\dot{V}\right)d\tau - \frac{e^{-i\omega_2 t}}{\omega_1 - \omega_2}\int_0^t e^{-i\omega_2 \tau}\left(i\frac{H}{AR}V + \frac{Pl}{Ag}\dot{V}\right)d\tau \quad [13]$$

$$\vartheta = ce^{-i\omega t} - \frac{u}{\omega}\cos\varphi - e^{-i\omega t}\int_0^t e^{-i\omega\tau}\left(\frac{i}{R}V + \frac{\omega}{g}\dot{V}\right)d\tau \quad [14]$$

Here

$$\omega = \frac{Pl}{H}$$

$$\omega_1 = \frac{H - \sqrt{H^2 + 4PlA}}{2A}$$

$$\omega_2 = \frac{H + \sqrt{H^2 + 4PlA}}{2A} \quad [15]$$

The first two terms in formula [13] define the precessional and nutational variations in the gyroscope axis as a function of the initial deviations. The third term represents the deviation of the gyroscope axis from the vertical as a result of the earth's rotation. The two latter terms compensate for the deviation of the gyroscope pendulum caused by the motion of the suspension point. The deviation is accounted for by the last term in Eq. [14]. By disregarding the terms that are not relevant to our purpose, we obtain the difference

$$\epsilon = \theta - \vartheta$$

$$\epsilon = \frac{e^{i\omega_2 t}}{\omega_1 - \omega_2}\int_0^t e^{-i\omega_1\tau}\left(i\frac{H}{AR}V + \frac{Pl}{Ag}\dot{V}\right)d\tau - \frac{e^{i\omega_2 t}}{\omega_1 - \omega_2}\int_0^t e^{-i\omega_2\tau}\left(i\frac{H}{AR} + \frac{Pl}{Ag}\right)d\tau + e^{-i\omega t}\int_0^t e^{-i\omega t}\left(\frac{i}{R}V + \frac{\omega}{g}\dot{V}\right)d\tau \quad [16]$$

It follows from formula [15] that when H is large enough

$$\omega_1 \approx -\frac{Pl}{H} = -\omega$$

$$\omega_2 \approx \frac{H}{A} \quad [17]$$

$$\omega_1 - \omega_2 \approx -\frac{H}{A}$$

Consequently, approximately

$$\epsilon = e^{-i\omega_2 t}\int_0^t e^{-i\omega_2\tau}\left(\frac{i}{R}V + \frac{\omega}{g}\dot{V}\right)d\tau \quad [18]$$

Let us consider some special cases.

a) Let the suspension point of a gyroscopic pendulum move along a steady course at constant velocity. Then

$$V = V_0 + Wt$$

Formula [18] gives

$$\epsilon = \left[\frac{1}{R}\left(\frac{V_0}{\omega_2} - i\frac{W}{\omega_2^2}\right) - i\frac{\omega}{g}\frac{W}{\omega_2}\right](e^{i\omega_2 t} - 1) - \frac{Wt}{R\omega_2} \quad [19]$$

Taking [17] into account, we can conclude that, in this case

$$\epsilon = O\left(\frac{1}{H}\right)$$

By the same token, it is apparent that the deviation itself, determined by formula [14], will be of an order of magnitude of H.

b) The point of support moves along the radius of a great circle, carrying out harmonic oscillations. Then, $V = \alpha \sin kt$ ($\mathrm{Im}\, k = 0$). Computations give

$$\epsilon = \frac{a}{k_2 - \omega_2^2}\left[\left(\frac{\omega_2}{R} + \frac{\omega k^2}{g}\right)\sin kt - ik\left(\frac{1}{R} - \frac{\omega\omega_2}{g}\right)\cos kt + ik\left(\frac{1}{R} + \frac{\omega\omega_2}{g}\right)e^{i\omega_2 t} \quad [20]$$

and

$$\vartheta = \frac{a}{k^2 - \omega^2}\left[\left(\frac{\omega}{R} - \frac{\omega k^2}{g}\right)\sin kt + ik\left(\frac{1}{R} - \frac{\omega^2}{g}\right)\cos kt - ik\left(\frac{1}{R} - \frac{\omega^2}{g}\right)e^{-i\omega t} \quad [21]$$

Hence, it follows that when k is close to value ω_2, the precessional theory equations cannot be used. If $\omega_2 \gg k \gg \omega$, then, as is apparent, the expansion of ϵ in a series of powers of $1/H$ beginning with the terms $1/H$, since the expansion of ϑ begins from the term $(1/H)^\circ$. Therefore, the precessional theory equations are applicable to cases of this type.

Let us assume that the requirement for Schuler's theorem has been fulfilled, i.e.

$$\frac{1}{R} = \frac{\omega^2}{g}$$

Then

$$\vartheta = -\frac{Ha}{PlR}\sin kt \quad [22]$$

while

$$\epsilon = \frac{a}{k^2 - \omega_2^2}\left[\frac{\omega}{g}(\omega\omega_2 + k^2)\sin k - i\frac{k\omega}{g}(\omega + \omega_2)\cos kt + i\frac{k\omega}{g}(\omega + \omega_2)e^{i\omega_2 t}\right]$$

Under these conditions, on the basis of formula [14], we obtain

$$\epsilon = O\left(\frac{1}{H^2}\right)$$

c) The suspension point moves along a circumference of radius ρ with a constant velocity v_0. Then

$$v = v_\theta r^{iv_0 t/\rho}$$

From formula [18], we have

$$\epsilon = \frac{\dfrac{1}{R} + \dfrac{\omega v_0}{g\rho}}{\dfrac{v_0}{\rho} - \omega_2}(e^{iv_0 t/\rho} - e^{i\omega_2 t}) \quad [23]$$

Hence, it follows that $\epsilon = 0(1/H)$, while, as can readily be seen, the expansion of ϑ begins with zero order terms relative to $(1/H)$. Hence the precessional theory equation is sufficiently accurate if we exclude the rare case of $v_0/\rho = \omega_2$.

d) Let us now assume that v and its derivatives of any order are limited by the modulus on the interval $[0, t]$:

$$\left|\frac{d^k v}{dt^k}\right| < N \qquad (k = 0, 1, 2, \ldots) \quad [24]$$

Repeating the integration by parts in formula [18], we obtain

$$\epsilon = \left\{\frac{i}{R}\left[\frac{v_0}{i\omega_2} + \frac{v_0}{(i\omega_2)^2} + \ldots\right] + \frac{\omega}{g}\left[\frac{\dot{v}_0}{i\omega_2} + \frac{\ddot{v}_0}{(i\omega_2)^2} + \ldots\right]\right\} e^{i\omega_2 t} - \left\{\frac{i}{R}\left[\frac{v}{i\omega_2} + \frac{\dot{v}}{(i\omega_2)^2} + \ldots\right] - \frac{\omega}{g}\left[\frac{\dot{v}}{i\omega_2} + \frac{\ddot{v}}{(i\omega_2)^2} + \ldots\right]\right\} \quad [25]$$

When condition [24] holds, the series in [25] converge absolutely. For a sufficiently large H, they converge at least as a geometric progression. Thus, in this case as well

$$\epsilon = O\left(\frac{1}{H}\right)$$

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Reviewer's Comment

The total angular momentum of a precessing gyroscope about its instantaneous axis of rotation is the vector sum of the angular momenta about the spin axis and about the precession axis. In this paper, the author investigates the motion of a gyroscopic pendulum on the assumption that the total angular momentum is about the spin axis and compares the result with that obtained for the case where not all the angular momentum is assumed to be about the spin axis. The two assumptions lead to the systems of differential equations [5] and [6]. These systems are reduced to the single equations [8] and [9] by means of the complex variables [7].

The application of [8] and [9] to the gyroscopic pendulum by means of [10] leads to Eqs. [11] and [12], the solutions of which are [13] and [14], respectively. The difference ϵ between the solutions of [13] and [14] is a measure of the lack of agreement in the behavior of the gyroscopic pendulum in the two cases considered. Under the simplifying assumptions [17], the value of ϵ is given by Eq. [18].

The author then applies [18] to the motion of the gyroscopic pendulum when the point of suspension moves in three different manners and finds that in one case the equations based on Eqs. [2] cannot be used.

The results in this paper seem to be new.

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Calendar of Forthcoming Soviet Bloc Meetings

Date	Meeting	Location
1963	Heat and Mass Transfer Conference	Minsk, USSR
1963	3rd Conference on High Energy Particle Accelerators and Instrumentation (International Union of Pure and Applied Physics)	Moscow, USSR
1963	Conference on Physics of Nuclear Fission	USSR
1963	5th All-Union Seminar on Refractory Metals	USSR
March 1963	USA-USSR Joint Symposium on Partial Differential Equations	Novosibirsk, USSR
March 1963	3rd Seminar on Titanium Alloys	Moscow, USSR
April-May 1963	2nd Conference on Program-Controlled Machines	Kiev, USSR
May 1963	2nd Optical Conference (Hungarian Society for Optics, Acoustics, and Cinematography)	Budapest, Hungary
June 1963	Space Research Committee Meeting (COSPAR) (International Council of Scientific Unions)	Warsaw, Poland
Aug. 27–31, 1963	4th Conference on Scientific and Applied Photography (Hungarian Society for Optics, Acoustics, and Cinematography)	Budapest, Hungary
Sept. 2–5, 1963	Symposium on Non-Classical Shell Problems (International Association for Shell Structures)	Warsaw, Poland
Sept. 25–27, 1963	Nitro-Compounds Symposium (International Union of Pure and Applied Chemistry)	Warsaw, Poland
1964	Conference on Cavitation and Corrosion-Abrasive Destruction	Moscow, USSR
1964	3rd Conference on Precision in Machine Building	Moscow, USSR
1964	2nd All-Union Congress on Theoretical and Applied Mechanics	Moscow, USSR
1964	3rd National Conference on Electronics	Prague, Czechoslovakia
1964	7th International Congress on High Speed Photography	Moscow, USSR
1964	12th Conference on High Energy Nuclear Physics (International Union of Pure and Applied Physics)	USSR
May-June 1964	3rd Assembly, Telegraph and Telephone Consultative Committee (International Telecommunications Union)	Moscow, USSR
June 1964	18th Annual Meeting of the International Union of Testing and Research Laboratories for Materials and Structures	Moscow, USSR
Fall 1964	4th Congress of the International Council of Aeronautical Sciences	Warsaw, Poland
Fall 1964	15th Congress of the International Astronautical Federation	Warsaw, Poland
1967	6th Conference of the International Federation of Medical Electronics	Moscow, USSR

Many of the above dates and locations are subject to change. For further information regarding attendance at and preparation for scientific and technical conferences in these countries, please direct inquiries to the National Academy of Sciences, 2101 Constitution Avenue, N. W., Washington 25, D. C., or to the Department of State, Soviet and Eastern European Exchanges Staff, Washington 25, D. C.